Thermodynamic Product Formula for Taub-NUT Black Hole

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Abstract

We derive various important thermodynamic relations of the inner and outer horizon in the background of Taub-NUT(Newman-Unti-Tamburino) black hole in four dimensional *Lorentzian geometry*. We compare these properties with the properties of Reissner Nordstrøm black hole. We compute *area product, area sum, area minus and area division* of black hole horizons. We show that they all are not universal quantities. Based on these relations, we compute the area bound of all horizons. From area bound, we derive entropy bound and irreducible mass bound for both the horizons. We further study the stability of such black hole by computing the specific heat for both the horizons. It is shown that due to negative specific heat the black hole is thermodynamically unstable. All these calculations might be helpful to understanding the nature of black hole entropy both *interior* and exterior at the microscopic level.

1 Introduction

There has recently been intense interest in the thermodynamic product formulae in both general relativity community[1] and string/M-theory community[2] to understand the black hole(BH) entropy[3] of multi-horizons at the microscopic level. Significant achievements have been made in case of asymptotically flat super-symmetric BHs in four and five dimension, where the microscopic degrees of freedom could be explained in terms of a two-dimensional(2D) conformal field theory(CFT)[4]. For microscopic entropy of extreme rotating BH the results could be found in [5].

In case of a regular axisymmetric and stationary spacetime of Einstein-Maxwell gravity with surrounding matter, the area product formula of event horizon($\mathcal{H}^+$) and Cauchy horizons($\mathcal{H}^-$) is[1]

$$\frac{\mathcal{A}_+\mathcal{A}_-}{64\pi^2} = J^2 + \frac{Q^4}{4} . \tag{1}$$

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and consequently the entropy product formula of $\mathcal{H}^{\pm}$ is

$$\frac{\mathcal{S}_+\mathcal{S}_-}{4\pi^2} = J^2 + \frac{Q^4}{4} . \tag{2}$$

In the absence of Maxwell gravity, these product formulae reduces to the following form[8]:

$$\frac{\mathcal{A}_+\mathcal{A}_-}{64\pi^2} = J^2 . \tag{3}$$

and

$$\frac{\mathcal{S}_+\mathcal{S}_-}{4\pi^2} = J^2 . \tag{4}$$

In the above formulae, the interesting point is that they all are independent of the mass, so-called the ADM(Arnowitt-Deser-Misner) mass of the back-ground space-time. Thus they all are universal quantities in this sense. Now if we incorporate the BPS states, the area product formula should be[2]

$$\frac{\mathcal{A}_+\mathcal{A}_-}{64\pi^2} = \left(\sqrt{N_1} \pm \sqrt{N_2}\right) = N, \ N \in \mathbb{N}, N_1 \in \mathbb{N}, N_2 \in \mathbb{N} . \tag{5}$$

where the integers N_1 and N_2 may be viewed as the excitation numbers of the left and right moving modes of a weakly-coupled two-dimensional CFT. N_1 and N_2 are depends explicitly on all the BH parameters.

Consequently, the entropy product formula of $\mathcal{H}^{\pm}$ should be

$$\frac{\mathcal{S}_+\mathcal{S}_-}{4\pi^2} = \left(\sqrt{N_1} \pm \sqrt{N_2}\right) = N, \ N \in \mathbb{N}, N_1 \in \mathbb{N}, N_2 \in \mathbb{N} . \tag{6}$$

It implies that the product of $\mathcal{H}^{\pm}$ could be expressed in terms of the underlying CFT which can be also interpreted in terms of a level matching condition. That means the product of entropy of $\mathcal{H}^{\pm}$ is an integer quantity[6].

It is true that certain BH possesses inner horizon or Cauchy horizon(CH) in addition to the outer horizon or event horizon. Thus there might be a relevance of inner horizon in BH thermodynamics to understanding the microscopic nature of inner BH entropy in comparison with the outer BH entropy. It is also true that CH is a blue-shift region whereas event horizon is a red-shift region by its own nature. Furthermore the CH is highly unstable due to the exterior perturbation[7]. Despite the above characteristics, the CH horizon is playing a crucial role in BH thermodynamics.

In this work, we shall focus on thermodynamic properties of both inner horizon and outer horizons of Lorentzian Taub-NUT BH. The special property of this BH is it is non-asymptotic type in comparison with RN BH, which is asymptotic type. The special features give a motivation to study them.

2

The plan of the paper is as follows. In Sec.(2), we describe various thermodynamic properties for Taub-NUT BH. In this Sec., there are five sub-section. In first sub-section, we derive the area bound of all horizons for Taub-NUT BH. In second sub-section, we derive the entropy bound for Taub-NUT BH. In third subsection, we compute the irreducible mass bound of $\mathcal{H}^{\pm}$. In fourth subsection, we derive temperature bound of all horizons and finally in last sub-section, we compute specific heat bound for both the horizons. Finally, we conclude in Sec.(3).

2 Thermodynamic properties of Lorentzian Taub-NUT BH:

The Lorentzian Taub-NUT(TN) BH is a stationary, spherically-symmetric vacuum solution of Einstein equations with the NUT parameter (n). The NUT charge or dual mass has an intrinsic feature in Einstein's general relativity and which is a gravitational analogue of a magnetic monopole in Maxwell's electrodynamics [9]. The presence of the NUT parameter in the space-time destroy its asymptotic structure making it, in contrast to the RN space-time, asymptotically non-flat.

The metric is given by [10, 11, 12]

$$ds^2 = -\mathcal{B}(r)\left(dt + 2n\cos\theta d\phi\right)^2 + \frac{dr^2}{\mathcal{B}(r)} + \left(r^2 + n^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{7}$$

$$\mathcal{B}(r) = 1 - \frac{2(\mathcal{M}r + n^2)}{r^2 + n^2} \tag{8}$$

where, $\mathcal{M}$ denotes the gravito-electric mass or ADM mass and n denotes the gravito-magnetic mass or dual mass or magnetic mass of the space-time. The idea of dual mass could be found in [13]. It is evident that there are two type of singularities are present in the metric (7). One type occurs at $B(r) = 0$ which gives us the Killing horizons or BH horizons:

$$r_{\pm} = \mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2} \text{ and } r_+ > r_- \tag{9}$$

r_+ is called event horizon $(\mathcal{H}^+)$ or outer horizon and r_- is called Cauchy horizon $(\mathcal{H}^-)$ or inner horizon. The other type of singularity occurs at $\theta = 0$ and $\theta = \pi$, where the determinant of the metric component vanishes. Misner[14] showed that in order to remove the apparent singularities at $\theta = 0$ and at $\theta = \pi$, t must be identified modulo $8\pi n$. Provided that $r^2 + n^2 \neq 2(\mathcal{M}r + n^2)$. It should be noted that the NUT parameter actually measures deviation from the asymptotic flatness at infinity which could be manifested in the off-diagonal components of the metric and this is happening due to presence of the Dirac-Misner type of singularity.

3

As long as

$$\mathcal{M}^2 + n^2 \geq 0 . \tag{10}$$

then the TN metric describes a BH, otherwise it has a naked singularity. When $\mathcal{M}^2 + n^2 = 0$, we find the extremal TN BH.

The product and sum of horizon radii becomes

$$r_+ r_- = -n^2 \ \text{ and } \ r_+ + r_- = 2\mathcal{M} . \tag{11}$$

The area[15] of this BH is given by

$$\mathcal{A}_\pm = \int_0^{2\pi} \int_0^\pi \sqrt{g_{\theta\theta} g_{\phi\phi}}|_{r=r_\pm} d\theta d\phi = 8\pi \left[(\mathcal{M}^2 + n^2) \pm \mathcal{M}\sqrt{\mathcal{M}^2 + n^2} \right] . \tag{12}$$

Their product[15] and sum yields

$$\mathcal{A}_+ \mathcal{A}_- = (8\pi n)^2 \left(\mathcal{M}^2 + n^2 \right) \ \text{ and } \ \mathcal{A}_+ + \mathcal{A}_- = 16\pi \left(\mathcal{M}^2 + n^2 \right) . \tag{13}$$

It indicates that the product is dependent on both mass parameter and NUT parameter. Thus it is not an universal quantity. Thus the conjecture "area product is universal" does not hold for non-asymptotic TN spacetime.

For completeness, we further compute the area minus and area division:

$$\mathcal{A}_\pm - \mathcal{A}_\mp = 8\pi \mathcal{M} T_\pm \mathcal{A}_\pm . \tag{14}$$

and

$$\frac{\mathcal{A}_+}{\mathcal{A}_-} = \frac{r_+^2 + n^2}{r_-^2 + n^2} . \tag{15}$$

Again, the sum of area inverse is found to be

$$\frac{1}{\mathcal{A}_+} + \frac{1}{\mathcal{A}_-} = \frac{1}{4\pi n^2} . \tag{16}$$

and the minus of area inverse is computed to be

$$\frac{1}{\mathcal{A}_\pm} - \frac{1}{\mathcal{A}_\mp} = \mp \frac{\mathcal{M}}{8\pi n^2 \sqrt{\mathcal{M}^2 + n^2}} . \tag{17}$$

It indicates that they are all mass dependent relations.

Likewise, the entropy product[15] and entropy sum of $\mathcal{H}^\pm$ becomes:

$$\mathcal{S}_- \mathcal{S}_+ = (2\pi n)^2 \left(\mathcal{M}^2 + n^2 \right) \ \text{ and } \ \mathcal{S}_- + \mathcal{S}_+ = 4\pi \left(\mathcal{M}^2 + n^2 \right) . \tag{18}$$

4

For our record, we also compute the entropy minus of $\mathcal{H}^{\pm}$ as

$$\mathcal{S}_{\pm} - \mathcal{S}_{\mp} = 8\pi\mathcal{M}T_{\pm}\mathcal{S}_{\pm} . \tag{19}$$

and the entropy division of $\mathcal{H}^{\pm}$ as

$$\frac{\mathcal{S}_+}{\mathcal{S}_-} = \frac{r_+^2 + n^2}{r_-^2 + n^2} . \tag{20}$$

Again, the sum of entropy inverse is found to be

$$\frac{1}{\mathcal{S}_+} + \frac{1}{\mathcal{S}_-} = \frac{1}{\pi n^2} . \tag{21}$$

and the minus of entropy inverse is

$$\frac{1}{\mathcal{S}_{\pm}} - \frac{1}{\mathcal{S}_{\mp}} = \mp\frac{\mathcal{M}}{2\pi n^2\sqrt{\mathcal{M}^2 + n^2}} . \tag{22}$$

The Hawking[16] temperature of $\mathcal{H}^{\pm}$ reads off

$$T_{\pm} = \frac{r_{\pm} - r_{\mp}}{8\pi\left(\mathcal{M}r_{\pm} + n^2\right)} \text{ and } T_+ > T_-. \tag{23}$$

Their product[15] and sum yields

$$T_+T_- = -\frac{1}{4\pi n^2} \text{ and } T_+ + T_- = -\frac{\mathcal{M}}{2\pi n^2}. \tag{24}$$

It may be noted that surface temperature product is independent of mass while sum depends on mass thus the product is universal whereas sum is not universal. It is also shown that for TN BH:

$$T_+\mathcal{S}_+ + T_-\mathcal{S}_- = 0. \tag{25}$$

It is clearly evident that it is a mass independent(universal) relation and implies that $T_+\mathcal{S}_+ = -T_-\mathcal{S}_-$ may be taken as a criterion whether there is a 2D CFT dual for the BHs in the Einstein gravity and other diffeomorphism gravity theories [17, 18]. This universal relation also indicates that the left and right central charges are equal i.e. $c_L = c_R$, which is holographically dual to 2D CFT.

2.1 Area Bound of TN BH for $\mathcal{H}^{\pm}$:

Now we are ready to derive the area bound relations for both the horizons. Using the ineqality Eq. (10) one can obtain $\mathcal{M}^2 \geq -n^2$. Since $r_+ \geq r_-$, one can obtain $\mathcal{A}_+ \geq \mathcal{A}_- \geq 0$. Then the area product gives:

$$\mathcal{A}_+ \geq \sqrt{\mathcal{A}_+\mathcal{A}_-} = 8\pi n\sqrt{\mathcal{M}^2 + n^2} \geq \mathcal{A}_- . \tag{26}$$

5

and the area sum gives:

$$16\pi \left(\mathcal{M}^2 + n^2\right) = \mathcal{A}_+ + \mathcal{A}_- \geq \mathcal{A}_+ \geq \frac{\mathcal{A}_+ + \mathcal{A}_-}{2} = 8\pi \left(\mathcal{M}^2 + n^2\right) \ . \tag{27}$$

Therefore the area bound for $\mathcal{H}^+$:

$$8\pi \left(\mathcal{M}^2 + n^2\right) \leq \mathcal{A}_+ \leq 16\pi \left(\mathcal{M}^2 + n^2\right) \ . \tag{28}$$

and the area bound for $\mathcal{H}^-$:

$$0 \leq \mathcal{A}_- \leq 8\pi n \sqrt{\mathcal{M}^2 + n^2} \ . \tag{29}$$

2.2 Entropy Bound for $\mathcal{H}^{\pm}$:

Similarly, since $r_+ \geq r_-$, one can get $\mathcal{S}_+ \geq \mathcal{S}_- \geq 0$. Therefore the entropy product gives:

$$\mathcal{S}_+ \geq \sqrt{\mathcal{S}_+ \mathcal{S}_-} = 2\pi n \sqrt{\mathcal{M}^2 + n^2} \geq \mathcal{S}_- \ . \tag{30}$$

and the entropy sum gives:

$$4\pi \left(\mathcal{M}^2 + n^2\right) = \mathcal{S}_+ + \mathcal{S}_- \geq \mathcal{S}_+ \geq \frac{\mathcal{S}_+ + \mathcal{S}_-}{2} = 2\pi \left(\mathcal{M}^2 + n^2\right) \ . \tag{31}$$

Thus the entropy bound for $\mathcal{H}^+$:

$$2\pi \left(\mathcal{M}^2 + n^2\right) \leq \mathcal{S}_+ \leq 4\pi \left(\mathcal{M}^2 + n^2\right) \ . \tag{32}$$

and the entropy bound for $\mathcal{H}^-$:

$$0 \leq \mathcal{S}_- \leq 2\pi n \sqrt{\mathcal{M}^2 + n^2} \ . \tag{33}$$

2.3 Irreducible Mass Bound for $\mathcal{H}^{\pm}$:

Christodoulou[19] had derived a relation between surface area of the $\mathcal{H}^+$ and irreducible mass, which could be written as

$$\mathcal{M}_{\text{irr},\pm} \ = \ \sqrt{\frac{\mathcal{A}_\pm}{16\pi}} = \sqrt{\frac{\mathcal{S}_\pm}{4\pi}} \ . \tag{34}$$

Now the product and sum of the irreducible mass for both the horizons are

$$\mathcal{M}_{\text{irr},+} \mathcal{M}_{\text{irr},-} \ = \ \frac{n\sqrt{\mathcal{M}^2 + n^2}}{2} \ \text{and} \ \mathcal{M}^2_{\text{irr},+} + \mathcal{M}^2_{\text{irr},-} = \mathcal{M}^2 + n^2. \tag{35}$$

From the area bound, we get the irreducible mass bound for TN BH.

6

For $\mathcal{H}^+$:

$$\frac{\sqrt{\mathcal{M}^2 + n^2}}{\sqrt{2}} \leq \mathcal{M}_{irr,+} \leq \sqrt{\mathcal{M}^2 + n^2} \ . \tag{36}$$

and for $\mathcal{H}^-$:

$$0 \leq \mathcal{M}_{irr,-} \leq \frac{[n^2(\mathcal{M}^2 + n^2)]^{\frac{1}{4}}}{\sqrt{2}} \ . \tag{37}$$

Eq. 36 is nothing but the Penrose inequality, which is the first geometric inequality for BHs[20].

2.4   Temperature Bound for $\mathcal{H}^\pm$:

In BH thermodynamics, temperature is an important thermodynamic parameter. So there must exists temperature bound relation on the horizons. As is when $r_+ \geq r_-$, one must obtain $\mid T_+ \mid \geq \mid T_- \mid \geq 0$.

Therefore the temperature product gives:

$$|T_+| \geq \sqrt{|T_+ T_-|} = \frac{1}{4\pi n} \geq |T_-| \ . \tag{38}$$

and the temperature sum gives:

$$\frac{\mathcal{M}}{2\pi n^2} = |T_+ + T_-| \geq |T_+| \geq \frac{|T_+ + T_-|}{2} = \frac{\mathcal{M}}{4\pi n^2} \ . \tag{39}$$

Thus the temperature bound for $\mathcal{H}^+$:

$$\frac{\mathcal{M}}{4\pi n^2} \leq |T_+| \leq \frac{\mathcal{M}}{2\pi n^2} \ . \tag{40}$$

and the temperature bound for $\mathcal{H}^-$:

$$0 \leq |T_-| \leq \frac{1}{4\pi n} \ . \tag{41}$$

2.5   Bound on Heat Capacity $C_\pm$ for $\mathcal{H}^\pm$:

The specific heat can be defined as

$$C_\pm = \frac{\partial \mathcal{M}}{\partial T_\pm}. \tag{42}$$

After some calculations, we obtain the specific heat for both the horizons:

$$C_\pm = -2\pi(r_\pm^2 + n^2). \tag{43}$$

7

Their product and sum on $\mathcal{H}^{\pm}$ yields:

$$C_+ C_- \;=\; (4\pi n)^2 \left(\mathcal{M}^2 + n^2\right). \tag{44}$$

and

$$C_+ + C_- \;=\; -8\pi \left(\mathcal{M}^2 + n^2\right). \tag{45}$$

Using $\mathcal{M}^2 \geq -n^2$ with the product of heat capacity and the sum of heat capacity, we get the bound on heat capacity for both the horizons. For $\mathcal{H}^+$:

$$4\pi \left(\mathcal{M}^2 + n^2\right) \leq |C_+| \leq 8\pi \left(\mathcal{M}^2 + n^2\right)\ . \tag{46}$$

and for $\mathcal{H}^-$:

$$0 \leq |C_-| \leq 4\pi n \sqrt{\mathcal{M}^2 + n^2}\ . \tag{47}$$

Using Eq. 13, Eq. 18 and Eq. 44, we find a new relation among specific heat product, area product and entropy product which is given by

$$C_+ C_- \;=\; \frac{\mathcal{A}_+ \mathcal{A}_-}{4} = 4\mathcal{S}_+ \mathcal{S}_-. \tag{48}$$

So far we have calculated different thermodynamic quantities and these formulae might be useful to further understanding the microscopic nature of BH entropy both exterior and interior. Again, the entropy product of inner and outer horizons could be used to determine whether the classical BH entropy could be written as a Cardy formula, giving some evidence for a holographic description of BH/CFT correspondence[18, 21]. The above thermodynamic properties including the Hawking temperature and area of both the horizons may therefore be expected to play a crucial role in understanding the BH entropy of $\mathcal{H}^{\pm}$ at the microscopic level.

3   Conclusion:

In order to understand the black hole entropy at the microscopic level, we have studied both the inner horizon and outer horizon thermodynamics for TN BH in four dimensional Lorentzian geometry. We also computed various thermodynamic relations like product, sum, substraction and division of inner and outer horizons. Due to the presence of the NUT parameter, we have found that all the thermodynamic relations are mass dependent that means they are not universal.

Using the proposed relations, we also compute different thermodynamic bounds. Like area bound, entropy bound, temperature bound etc. We also derived a relation between specific heat product, area product and entropy product. The thermodynamic relations

that we have derived in this note must have an implications. We suggest these thermodynamic formulae gives further clue on understanding the microscopic nature of BH entropy (both inner and outer) for both the horizons. In [22], we derived thermodynamic product formula for another non-asymptotic type of BH like Kehagias-Sfetsos BH in Horava-Liftshitz gravity, where the area product is universal in nature.

Appendix:

In the appendix, we have given various thermodynamic relations for TN BH in comparison with RN BH.

Parameter$(i = +, -)$	RN BH	Taub-NUT BH
$r_{\pm}$:	$\mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2}$	$\mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2}$
$\sum r_i$:	$2\mathcal{M}$	$2\mathcal{M}$
$\prod r_i$:	Q^2	$-n^2$
$\mathcal{A}_{\pm}$:	$4\pi\left(2\mathcal{M}r_{\pm} - Q^2\right)$	$8\pi\left(\mathcal{M}r_{\pm} + n^2\right)$
$\sum \mathcal{A}_i$:	$8\pi\left(2\mathcal{M}^2 - Q^2\right)$	$16\pi\left(\mathcal{M}^2 + n^2\right)$
$\prod \mathcal{A}_i$:	$(4\pi Q^2)^2$	$(8\pi n)^2\left[\mathcal{M}^2 + n^2\right]$
$\mathcal{S}_{\pm}$:	$\pi\left(2\mathcal{M}r_{\pm} - Q^2\right)$	$2\pi\left(\mathcal{M}r_{\pm} + n^2\right)$
$\sum \mathcal{S}_i$:	$2\pi(2\mathcal{M}^2 - Q^2)$	$4\pi\left(\mathcal{M}^2 + n^2\right)$
$\prod \mathcal{S}_i$:	$\pi^2 Q^4$	$(2\pi n)^2\left[\mathcal{M}^2 + n^2\right]$
$\kappa_{\pm}$:	$\frac{r_{\pm} - r_{\mp}}{2(2\mathcal{M}r_{\pm} - Q^2)}$	$\frac{r_{\pm} - r_{\mp}}{4(\mathcal{M}r_{\pm} + n^2)}$
$\sum \kappa_i$:	$\frac{4\mathcal{M}(Q^2 - \mathcal{M}^2)}{Q^4}$	$-\frac{\mathcal{M}}{n^2}$
$\prod \kappa_i$:	$\frac{Q^2 - \mathcal{M}^2}{Q^4}$	$-\frac{1}{4n^2}$
$T_{\pm}$:	$\frac{r_{\pm} - r_{\mp}}{4\pi r_{\pm}^2}$	$\frac{r_{\pm} - r_{\mp}}{4\pi(r_{\pm}^2 + n^2)}$
$\sum T_i$:	$\frac{2\mathcal{M}(Q^2 - \mathcal{M}^2)}{\pi Q^4}$	$-\frac{\mathcal{M}}{2\pi n^2}$
$\prod T_i$:	$\frac{(Q^2 - \mathcal{M}^2)}{4\pi^2(Q^4)}$	$-\frac{1}{(4\pi n)^2}$
$\mathcal{M}_{irr,\pm}$:	$\sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}}$	$\sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}}$
$\sum \mathcal{M}_{irr}^2$:	$\mathcal{M}^2 - \frac{Q^2}{2}$	$\mathcal{M}^2 + n^2$
$\prod \mathcal{M}_{irr}$:	$\frac{Q^2}{4}$	$\sqrt{\frac{n^2(\mathcal{M}^2 + n^2)}{4}}$

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